Evolution of complex oscillations in a quasiperiodically forced chain

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We investigate complex dynamics along the chain of quasiperiodically forced circle maps. We present numerical evidence for the development of a strange nonchaotic behavior within a wide range of spatial parameters. The bifurcations and properties of nontrivial attracting sets are studied. [S1063-651X(97)07912-9]

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I. INTRODUCTION

Much in the spirit of the study of the routes to chaos in finite-dimensional dynamical systems, investigation of the transition to turbulence in simple but pithy models of spatially extended systems is not only an interesting topic in and of itself, but may also provide *a posteriori* insights into the problem of the ''nature of turbulence.'' Toward this goal, modeling of a continuous medium by a chain of coupled oscillators, and characterization of complex phenomena in space-time, are important in the study of turbulence in a general sense not only in the field of hydrodynamics but also in chemical reactions, nonlinear optics, and biology.

A variety of phenomena have been discovered in lattices of coupled discrete- or continuous-time systems demonstrating a period-doubling transition to chaos; among them are spatiotemporal chaos, spatial period-doubling bifurcations, saturation of attractor dimension down the flow, critical dynamics, and development of multistable regimes [1–6].

In high-dimensional dynamical systems, chaos evolves through different scenarios. One of the typical routes to chaos, well known from theoretical results as well as from experiments, is the transition from quasiperiodic oscillations to a strange chaotic attractor [7–14].

Recently, considerable interest has been focused on the study of quasiperiodically forced nonlinear dynamical systems. A strange nonchaotic attractor (SNA) is one of the nontrivial attracting sets that can be observed in such systems. This attractor was first described by Grebogi *et al.* [15], and since then investigated in a number of numerical and experimental studies. Its properties are in between order and chaos: it has no exponential divergence of trajectories, but it is not a finite set of points and is not piecewise differentiable.

In this paper, we focus on the question: How do attractors with a complex structure develop along a chain of nonlinear dynamical systems when quasiperiodic forcing is applied? Hence, our investigations are made under the following strategy: proposition of a simple and relevant model (Sec. II); search and exploration of different routes from quasiperiodicity to chaos, qualitative description of the change of oscillatory properties along the chain, and verification of the existence of SNA in the spatially extended model (Sec. III); summary and a definition of future problems (Sec. IV).

II. CHAIN OF CIRCLE MAPS

In studying the quasiperiodic transition to chaos, it is convenient to investigate the circle map, which can be thought of as representing the Poincarè map of a continuous flow, and is also highly relevant for understanding many physical phenomena. We focus on the properties of the chain of circle maps that take the forms

$$x_0(n+1)=x_0(n)+\theta_0, \quad \text{mod } 1, \tag{1}$$

$$x_1(n+1)=x_1(n)+\Omega_1-(K_1/2\pi)\sin[2\pi x_1(n)] + A\cos[2\pi x_0(n)], \ \text{mod } 1, \tag{2}$$

$$x_j(n+1)=x_j(n)+\Omega_j-(K_j/2\pi)\sin[2\pi x_j(n)] + G\cos[2\pi x_{j-1}(n)], \quad \text{mod } 1, \tag{3}$$

where $j=2,3,\ldots,m$, and the parameters Ω and K are the phase shift and nonlinearity in the circle map, respectively.

Equation (1) describes an external forcing which can be considered as a signal from a similar oscillator but with $K=0$ and the fixed winding number θ_0 irrational. In circle map (2), a quasiperiodic external forcing with amplitude A is included. The dynamics of x_0 is added to the dynamics of x_1. The detailed numerical analysis of the system of equations (1) and (2) for $A\neq 0$ was performed in Refs. [16,17]. The existence of two- and three-frequency tori, and strange nonchaotic and chaotic attractors has been found. System (3) may be considered as a chain of coupled circle maps where the parameter G represents the coupling coefficient and cells are supposed to be identical, i.e., $\Omega_j=\Omega$, $K_j=K$. Furthermore, systems (1), (2), and (3) are the quasiperiodically forced chain of circle maps which is the subject of our interest.

III. COMPLEX DYNAMICS ALONG THE CHAIN

Adding a quasiperiodic forcing leads to a few notable effects. In this section we present our numerical work supporting this and other associated results, and illustrate the distinct characteristic properties of different attractors that can arise in quasiperiodically forced chains. The most important effect is that quasiperiodic forcing interrupts the sequence of spatial period-doubling bifurcations: instead of this, an invariant curve becomes fractal. The latter is addressed in detail below.

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A. Strange nonchaotic behavior

Up to now several scenarios of transition to chaos in dynamical systems observed under the change of some control parameters have been revealed and studied in detail. In dissipative systems there are the Feigenbaum period-doubling sequence, and transitions via intermittency and via quasiperiodicity. For $K>1$, the circle map is no longer invertible, and has chaotic trajectories. In this region for $A=0$ the development to chaos along the chain via a finite number of spatial period-doubling bifurcations takes place (Fig. 1) in contrast to an infinite number of period doublings in each cell [3,5]. In Fig. 1(a) it is clearly seen that chaotic oscillations develop in the fifth cell. Its Lyapunov exponent is equal to $\lambda=+0.0434$. The next cells add no positive exponents; they seem to retranslate chaotic dynamics from the fifth cell.

We now discuss, what qualitative changes appear when a quasiperiodic forcing with $\theta_0=0.5(\sqrt{5}-1)$ is applied . Quasiperiodic forcing leads to the interruption of a sequence of spatial period doublings as soon as the forcing amplitude is increased, and to the loss of smoothness of an invariant curve corresponding to a two-frequency quasiperiodic regime in terms of continuous-time systems. The sequence of plots, shown in Fig. 1(b), is representative of the transition from quasiperiodic dynamics to strange nonchaotic dynamics indicated in the fifth cell ($\lambda=-0.0516$).

To explain the main results, we refer to the bifurcation diagram shown in Fig. 2. According to the qualitatively different dynamical behavior exhibited by systems (1), (2), and (3), Fig. 2 can be divided into three regions. The white-color region corresponds to two-frequency quasiperiodic oscillations with two or four bands, respectively. The hatched area in the diagram indicates the set of parameter values for which the system still exhibits negative Lyapunov exponents, but the attractor becomes strange. To indicate the strangeness of the attractor in each cell we use the criteria described in Ref. [18]. Let us examine the appearance of an n-band strange nonchaotic attracting set when the spatial coordinate j is changed. For some value of A the transition from a four-band torus to a two-band SNA occurs (the solid curve in Fig. 2), being similar to merging crisis of n-band torus [17,19] or tori coexisting in the phase space [20]. Since the spatial coordinate is discrete, it is impossible to indicate the bifurcation of the band merging. Therefore, we can only refer to the analogy between these phenomena in time-space. The dashed curve in Fig. 2 illustrates the transition when the loss of smoothness of a two-band torus leads to a two-band fractal set. Notice that quasiperiodic forcing can not only suppress chaos, as mentioned above, but also induce the development of chaotic oscillations along the chain for larger values of the forcing amplitude.

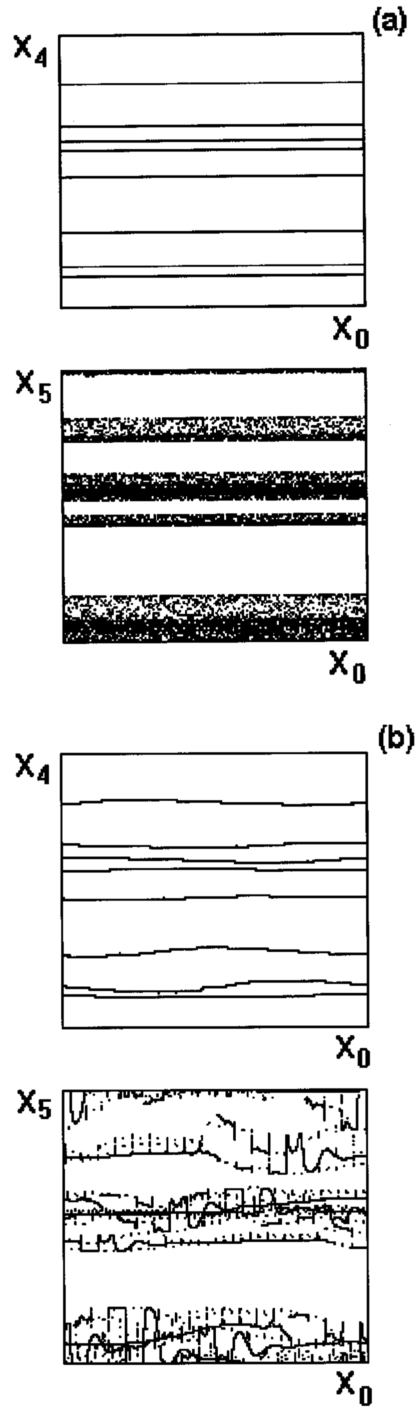

FIG. 1. The view of the attractors of systems (1), (2), and (3) [$K=1.55$, $\Omega=0.5$, $G=0.13$, $\theta_0=0.5(\sqrt{5}-1)$] in the quasiperiodically unforced case $A=0$ [doubled invariant curve ($j=4$) and chaotic attractor ($j=5$)] (a), and when quasiperiodic forcing is applied $A=0.001$ [wrinkling of doubled invariant curve ($j=4$) and strange nonchaotic attractor ($j=5$)] (b).

B. Spatial dynamics

The dynamics in each cell differs from the one in the other cells, and becomes more complicated along the chain. From some value of j the dynamics is qualitatively the same in each cell; let us call this regime ''spatiostable.'' Consider spatiostable regimes when the parameters are changed. We investigate a chain of 900 cells for different values of K and G and fixed A with the same initial conditions. The bifurcation diagram of spatiostable regimes is given in Fig. 3. Inside region 1 there exists a cycle with period 2 in all cells beginning with some j. In this case a quasiperiodic excitation damps along the chain, and has no effect on the oscillations of cells whose dynamics in the autonomous case corresponds

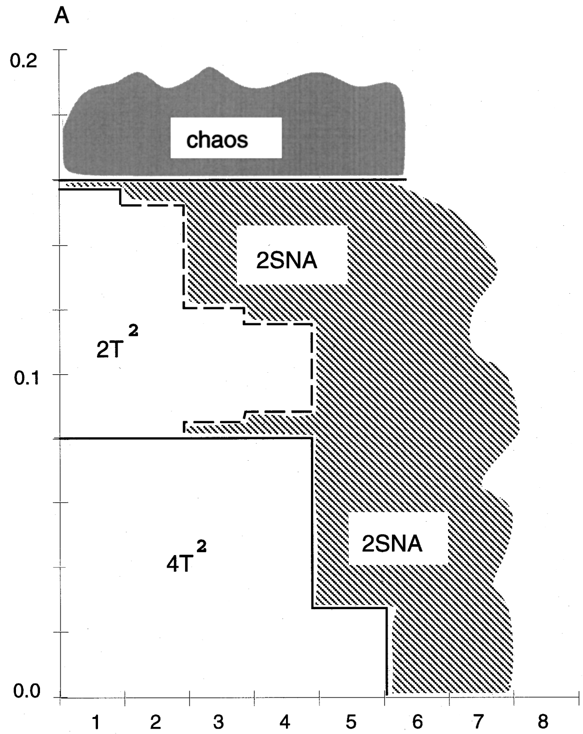

FIG. 2. Bifurcation diagram for a chain of coupled circle maps [$K=1.55$, $\Omega=0.5$, $G=0.1$, and $\theta_0=0.5\,(\sqrt{5}-1)$]. $2T^2$ is the two-band invariant curve, $4T^2$ is the four-band invariant curve, and 2SNA is the two-band strange nonchaotic attractor.

to a cycle with period 2. The cells oscillate ''in phase,'' and the instant states along the chain at a fixed time look like a straight line [Fig. 4(a)]. When the parameter nonlinearity K or the coupling parameter G is increased, the spatiostable regime 1 is destroyed. An external signal propagates along the chain, influences the dynamics, and gives rise to quasiperiodic oscillations in each cell. Corresponding instant states are shown in Fig. 4(b). In region 3, one of two different cycles with period 2 is realized in each cell along the chain [Fig. 4(c)]. It seems that the multistability phenomenon takes place, and the bifurcation diagram in Fig. 3 has a multisheet structure. The regime of SNA appears via the destruction of an invariant curve [Fig. 4(d)] and becomes spatiostable in regions 4 and 5 (Fig. 3).

When the parameter of nonlinearity is increased, chaotic oscillations appear (region 6 in Fig. 3). For different values of coupling chaotic oscillations have different properties in space. Below the dashed curve (Fig. 3), every other cell adds a positive Lyapunov exponent into the spectrum of Lyapunov exponents of the system. Hence the hyperchaos evolves along the chain. Figure 5 presents the results of calculation of Kolmogorov entropy ($H_j=\Sigma_{i=i}^{j}\lambda_i^+$) for different values of coupling G. It can be seen that, for $G<0.385$, the Kolmogorov entropy grows in a linearly way along the chain. But while G is increased, every other cell does not add positive Lyapunov exponents, and a saturation of the entropy takes place. Therefore, for chaotic oscillations developed from quasiperiodic ones, we obtained the same results as for chaos of the Rössler type [3].

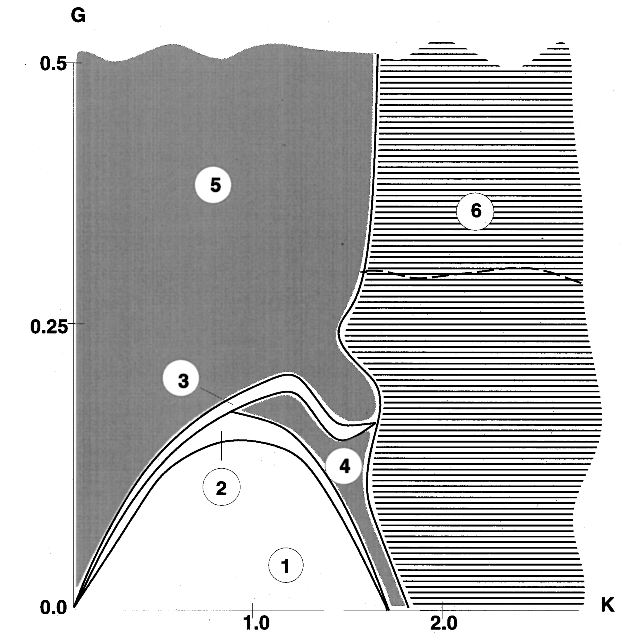

FIG. 3. Bifurcational diagram of spatiostable regimes in a quasiperiodic forced chain on a parameter plane (K,G) ($A=0.15$, and the initial conditions are homogeneous).

C. Attractors on a three-dimensional torus

Starting from a knowledge of the destruction of a two-dimensional torus [11,12], let us proceed to study the influence of the third frequency. For small nonlinearity ($K<1$), the circle map is invertible, and has no chaotic trajectories. In this case, the development of chaotic oscillations along the chain is impossible. However, the destruction of quasiperiodic oscillations and the transition to a strange nonchaotic attractor seem to be typical phenomena, being observed

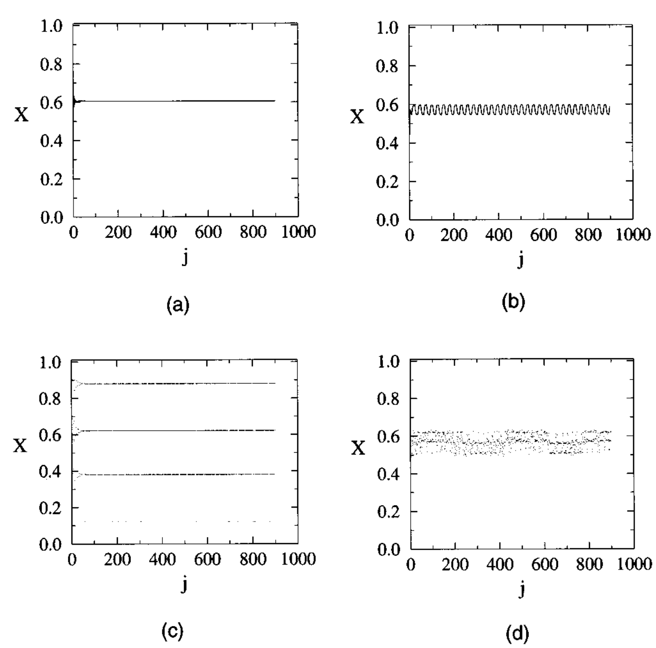

FIG. 4. Stable states along the chain at fixed time for different regions of the diagram in Fig. 3.

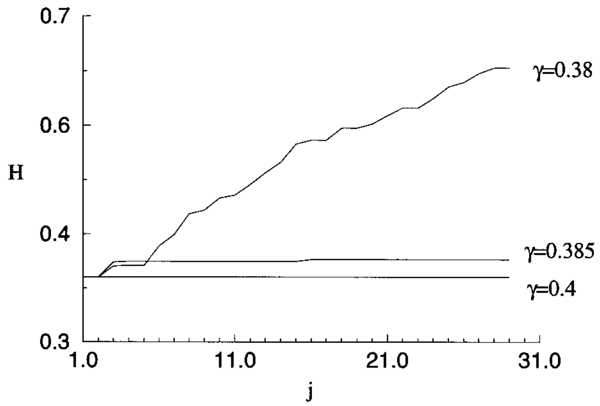

FIG. 5. Kolmogorov entropy vs the number of cells, for different G values [$K=2.0$, $\Omega=0.5$, $A=0.15$, and $\theta_0=0.5\ (\sqrt{5}-1)$].

within a wide range of cell parameters and coupling coefficient. Adding external forcing with frequency $\theta_0=\sqrt{2}-1$ (silver mean), and keeping the winding number in each cell at $\theta=0.5(\sqrt{5}-1)$ (golden mean), we provide the regime of three-frequency oscillations in terms of a continuous-time dynamical system (in terms of a map, it has two zero Lyapunov exponents, one of which is connected to the quasiperiodic forcing, and another one which is equal to 0.127×10^{-4}).

For some values of the forcing amplitude, synchronization on T^3 can take place, and the destruction of two-frequency synchronous oscillations along the chain discussed above occurs (Fig. 6). However, let us consider the evolution of the three-frequency quasiperiodic oscillations. A set of standard methods like phase portraits, power spectra, and Lyapunov exponents are used to distinguish different kinds of dynamics in systems exhibiting two-frequency quasiperiodic behavior. However, these quantities cannot sufficiently characterize three-frequency oscillations [Fig. 7(a)], in particular when it is necessary to decide whether a three-frequency quasiperiodic motion is broken down or not. For this reason the dimension of the limit set has been reduced using a Poincaré section. We have to formulate a condition which specifies the secant surface: $|x-0.5|\leqslant10^{-5}$. By means of this section we are able to obtain phase portraits of the system in the x_0-x_j plane [Fig. 7(b)], which can be compared with the phase portraits in Fig. 6. This yields an invariant curve in the case of an ergodic motion on T^3, and reveals its loss of smoothness and further destruction in the sixth cell. Thus the appearance of a strange nonchaotic attractor on T^3 is related to the destruction of this invariant curve in the Poincaré section. Thus, along the chain, the alternation of synchronous oscillations, three-frequency motion, and strange nonchaotic behavior can be distinguished.

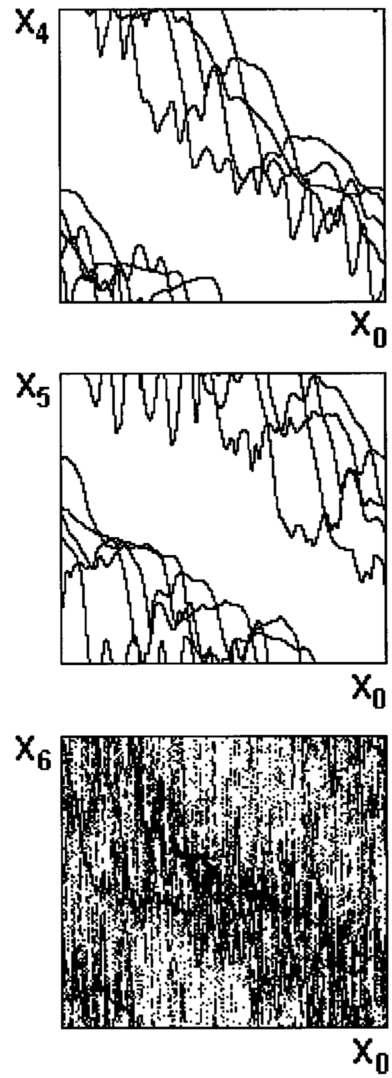

FIG. 6. Phase space plots corresponding to trajectories on a three-frequency quasiperiodic attractor when synchronization takes place ($K=0.8$, $\Omega=0.610\,074$, $G=0.2$, $A=0.06$, and $\theta_0=\sqrt{2}-1$). For $j=4$ and 5, an invariant curve is presented, while for $j=6$ a strange nonchaotic attractor is observed.

IV. CONCLUSIONS

It has been shown that the quasiperiodically forced chain of circle maps exhibits well-known phenomena investigated before, namely, spatial period doublings, multistability, and saturation of attractor dimension, as well as new effects related to the introduction of quasiperiodic forcing. One important feature is that a strange nonchaotic behavior is a typical spatial regime. Along the chain, it can be observed within a wide range of both spatial and coupling parameters, while in a single quasiperiodically driven map it is detected in a rather narrow region of the parameter space even if it exists over the set of positive measures in it.

Moreover, it is possible to introduce detuning between the winding numbers to obtain multifrequency quasiperiodic regimes and investigate the existence of attractors with a complex structure on their surface and their breakdown.

However, we realize that such an approach applied to maps on a torus has a limitation in the fact that the system studied there is a discrete one and is highly abstract, and all trajectories are on the torus by construction. Hence, a natural typical dissipative dynamical system which allows higher-dimensional phenomena to appear could be a challenge for further studies of the above-mentioned problems.

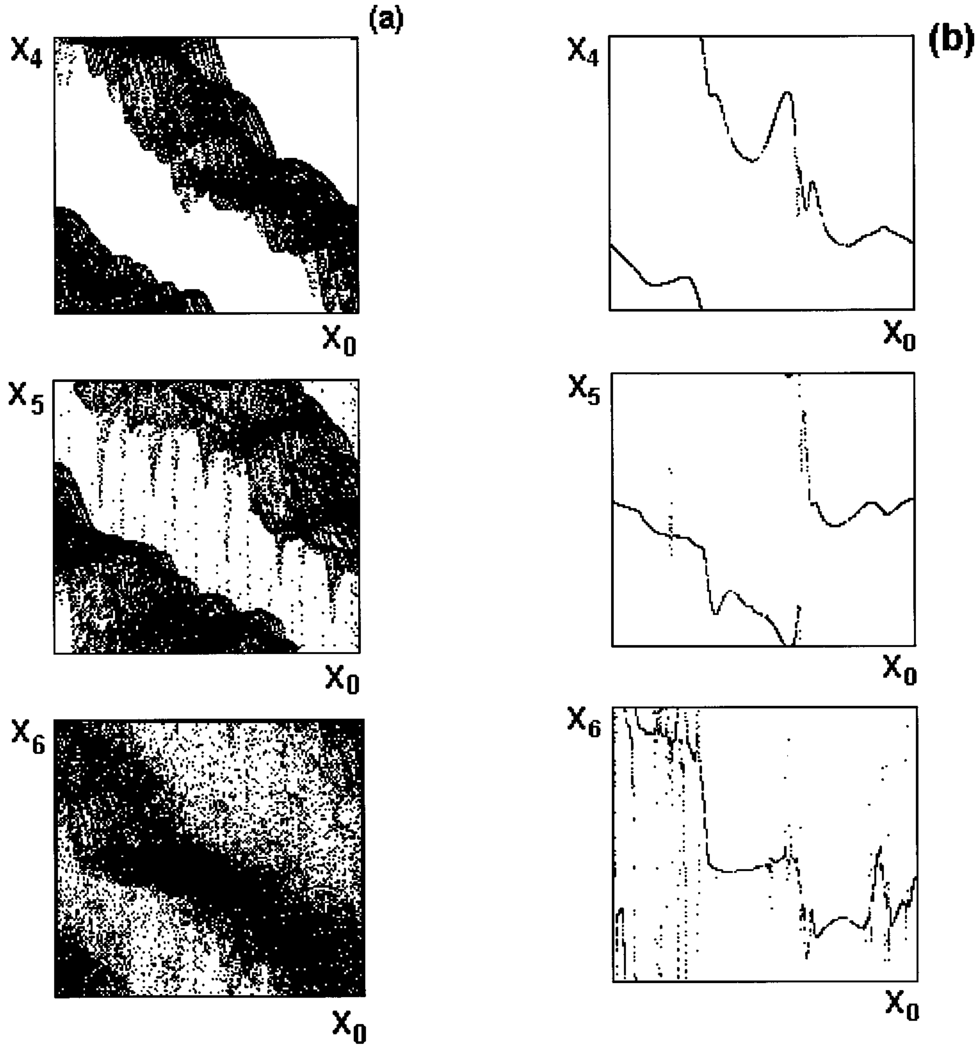

FIG. 7. Evolution of the phase portraits of a three-frequency invariant torus in systems (1), (2), and (3) ($K=0.8$, $\Omega=0.610\,074$, $G=0.2$, $A=0.058$, and $\theta_0=\sqrt{2}-1$) (a), and Poincaré maps (b) for attractors (a).

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[1] J. P. Crutchfield and K. Kaneko, *Phenomenology of Spatio-Temporal Chaos* (World Scientific, Singapore, 1987).
[2] K. Kaneko, Physica D **37**, 60 (1989).
[3] V. S. Anishchenko, D. E. Postnov, and M. A. Safonova, Pis'ma Zh. Tekh. Fiz. **11**, 1505 (1985) [Sov. Tech. Phys. Lett. **11**, 621 (1985)].
[4] I. S. Aranson, A. V. Gaponov-Grekhov, and M. I. Rabinovich, Physica D **33**, 1 (1988).
[5] S. P. Kuznetsov and A. S. Pikovsky, Physica D **19**, 384 (1986).
[6] S. P. Kuznetsov, Chaos Solitons Fractals **2**, 281 (1992).
[7] L. Landau, Dokl. Akad. Nauk USSR **44**, 339 (1944).
[8] E. A. Hopf, Commun. Pure Appl. Math. **1**, 303 (1948).
[9] D. Ruelle and F. Takens, Commun. Math. Phys. **20**, 167 (1971).
[10] S. Newhouse, D. Ruelle, and F. Takens, Commun. Math. Phys. **64**, 35 (1978).
[11] V. I. Arnol'd, V. S. Afraimovich, Yu. S. Iljashenko, and L. P. Shil'nikov, *Dynamical Systems. Vol 5. Bifurcation Theory and Catastrophe Theory* (Springer-Verlag, New York, 1994).
[12] V. S. Anishchenko, M. A. Safonova, U. Feudel, and J. Kurths, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 595 (1994).
[13] J. P. Gollub and S. V. Benson, J. Fluid Mech. **100**, 449 (1980).
[14] J. Stavans, F. Heslot, and A. Libchaber, Phys. Rev. Lett. **55**, 596 (1985).
[15] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Physica D **13**, 261 (1984).
[16] M. Ding, C. Grebogi, and E. Ott, Phys. Rev. A **39**, 2593 (1989).

[17] U. Feudel, J. Kurths, and A. Pikovsky, Physica D **88**, 176 (1995).
[18] A. Pikovsky and U. Feudel, Chaos **5**, 253 (1995).
[19] J. F. Heagy and S. M. Hammel, Physica D **70**, 140 (1994).
[20] V. S. Anishchenko, T. E. Vadivasova, and O. Sosnovtseva, Phys. Rev. E **53**, 4451 (1996).